Mathematical Concept of the Bloch Flow Equations for General Magnetic Resonance Imaging: A Review

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ABSTRACT: The recent analytical solutions to the Bloch NMR equations for a general RF excitation have opened many possibilities for further investigations to NMR theory and experiments even at the molecular level. Fortunately, many of the most important but hidden applications of blood flow and general physiological fluid flow parameters can be revealed without too much difficulty if appropriate mathematical techniques are used to explore the new NMR equations derived from the Bloch equations. Generally, we should be very much concerned with analytical results that the Bloch NMR flow equations can provide for different physical, biomedical, geophysical, medical, and environmental situations especially at the molecular level for the purpose of interdisciplinary approach to solve difficult problems. It can be motivating, exciting, and rewarding if attention are focused on the possible application of these analytical techniques and methods suitable for describing each of the various normal and pathological biological conditions. Most solutions presented in this study are described both in isotropic and anisotropic geometries with minimum mathematical assumptions. We discussed a general expression for the diffusion coefficients in the common geometries. These analytical results can prove to be very invaluable in the analysis of restricted flows. It is so much special because it could tell us when restricted flows occur and also reveal the causes of such restriction. Such knowledge can help in finding the causes of many diseases (whose causes are yet unknown) and suggest the best treatment for them. © 2011 Wiley Periodicals, Inc. Concepts Magn Reson Part A 38A: 85–101, 2011.

KEY WORDS: Bloch NMR flow equations; diffusion coefficient; NMR diffusion equation; magnetic resonance imaging (MRI)

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INTRODUCTION

Diffusion studies are valuable source of information on many facets of molecular organization and phase structure in restricted geometries. The knowledge of a variety of systems (such as sedimentary rocks, biological tissues, biomedical materials for clinical applications) can benefit from clarifying the role of water molecule diffusion in restricted geometries. Zeolites, for example are porous crystalline materials that absorb a number of molecules. Because of their property of high void volume, regular pore distribution, and shape selectivity, they are used extensively in industries as sieves and catalysts. Both the catalytic as well as sieving application of the zeolites depend on the diffusivities of the adsorbed molecules. Particularly, the cracking of large hydrocarbons inside the zeolitic pores is known to be determined by the ease with which the reactants and the product can diffuse in the pores which in turn depends on factors including the temperature, the shape and size of the pore and of adsorbed hydrocarbon, the concentration of the guest molecule, and so on. The sorption, binding, and the transport characteristics of various adsorbents in zeolitic pore system have been investigated extensively both experimentally and theoretically, with an objective to achieve an understanding of the behavior of guest molecule in confined geometries.

Biological tissues, on the other hand, are complex systems that contains a variety of liquid components macromolecules and ions. They are highly heterogeneous media that consists of various compartments and barriers of different diffusivities. In terms of its cyto-histologic architecture, a tissue can be regarded as a porous structure made up of a set of more or less connected compartments in a network—like arrangement. Because molecular motion of water is significantly affected by macromolecules, the variation in the relaxation times between tissues is attributed to the effect of macromolecular interaction. The movement of water molecules during diffusion-driven random displacement is restricted by compartmental boundaries and other molecular obstacle in such a way that the actual diffusion distance is reduced, compare with that expected in unrestricted diffusion. When diffusive properties change with the direction of diffusion, the prevailing condition is anisotropic and the associated displacement is no longer isotropic and Gaussian, like the unrestricted diffusion. An understanding of the detailed motion of molecules in a system by means of its diffusivities which has direct relationship with the movement of the spins and hence velocity of flowing tissue will be of tremendous help in understanding the physiology of the biological system.

In general, it is widely reported that the thermodynamic and transport properties of fluids are considered altered on their physical confinement in a well-defined channel and cavity system of porous materials - biological or nonbiological. Two competing effects seem to be the main contributors of the modification of the dynamics of fluid under confinement: i) The geometric confinement and ii) the interaction with the host cage. In other words, the problem can be addressed by asking how the properties of the porous medium, such as size, surface area, or the chemical nature of the interface can modify the dynamical behavior.

Flow and diffusion through porous media represent a vast field of study with many scientific and engineering applications (*1–58*). A detailed understanding of the complexities of flow and diffusion in porous materials is essential for the design, development, and optimization of catalysis and adsorption. (*59*). Diffusion and flow can be measured very delicately and accurately using an NMR system (*60*). The essence of the method is that motion of magnetic nuclei in a magnetic field and magnetic gradient results in those nuclei changing their Larmor precession frequency and their phase angle in the field. Because NMR can be set up to measure the number of nuclei at specific phase angles, the motion of groups of nuclei can be determined very accurately based on the fundamental Bloch NMR flow equations.

Due to its noninvasiveness, the NMR techniques have proved to be a powerful tool in studying flow in restricted geometries. It has been particularly useful for studying diffusion because they can provide accurate self-diffusion coefficient for the individual components or multi-components systems in a matter of minutes, whereas traditional radioactive tracer techniques may take weeks for each component and require isotropic substitution. There are two main ways in which NMR have been used to study self-diffusion coefficients: i) analysis of relaxation data and ii) pulsed field gradient (PFG) NMR (*61*). In the PFG method, the attenuation of a spin echo signal resulting from the dephasing of the nuclear spins due to the combination of the translational motion of the spins and the imposition of spatially well-defined gradient pulses is used to measure motion. In contradistinction to the relaxation method, no assumptions need to be made regarding the relaxation mechanisms or in relating to correlation time τ_c to the translational motion of the probe molecule. However, to determine the diffusion coefficient, D, as against

apparent diffusion coefficient, D_{app}, the effects of structural boundaries that affect the natural diffusion coefficient of the probe species needs to be considered. The application of gradients affords a powerful tool, not only for studying molecular diffusion under favorable circumstances down to ($<10^{17}m^2s^1$) but also for providing structural information in the range of 0.1 - 100μm when diffusion in restricted (e.g. diffusion in a cell) on the NMR time scale. Unfortunately, there is no single article that provides analytical options to the Bloch NMR flow equation from which one can adapt to solve specific problems (although there has been a good number of literature that have provided a very good discussion of the underlying principles (*61–65*)). Therefore, the differential equations in different geometries and coordinates derived from the Bloch NMR flow equations as assembled in this article can be taken as definitions of new functions to be studied in detail for specific needs.

MATHEMATICAL FORMULATION OF BLOCH EQUATIONS FOR DIFFUSION NMR

NMR is governed by the Bloch NMR flow equations, the equations which relate macroscopic concept of magnetization to the applied radiofrequency, gradient and static magnetic fields. The dynamics of the changes in bodies containing NMR—sensitive nuclei, its physical changes (for example, freely diffusing or bound within a cavity) are captured in NMR by the Bloch equations: the equations describing the physics of magnetic moments—such as the moment of the water proton—as a precessional, gyroscopic motion in the presence of exponential damping (T_1 and T_2), perturbing magnetic fields (the fixed B_0, and the time-varying radiofrequency B_1).

The Bloch NMR equations are a set of coupled differential equations describing the behavior of the macroscopic magnetization vector under any conditions. A form of the equations (*66–79*) is given as:

$$\frac{dM_x}{dt} = \frac{\partial M_x}{\partial t} + v\frac{\partial M_x}{\partial x} = -\frac{M_x}{T_2} \quad [1]$$

$$\frac{dM_y}{dt} = \frac{\partial M_y}{\partial t} + v\frac{\partial M_y}{\partial x} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \quad [2]$$

$$\frac{dM_z}{dt} = \frac{\partial M_z}{\partial t} + v\frac{\partial M_z}{\partial x} = -\gamma M_z B_1(x) - \frac{M_o - M_z}{T_1} \quad [3]$$

From the above equations, a partial differential equation of second order (*76*, *77*, *80*, *81*) is derived, which is very invaluable in the analyses of space and time dependence of the NMR transverse magnetization.

$$v^2\frac{\partial^2 M_y}{\partial x^2} + 2v\frac{\partial^2 M_y}{\partial x \partial t} + vT_o\frac{\partial M_y}{\partial x} + T_o\frac{\partial M_y}{\partial t} + \frac{\partial^2 M_y}{\partial t^2} + \left\{T_g + \gamma^2 B_1^2(x,t)\right\}M_y = F_o\gamma B_1(x,t) \quad [4]$$

where $T_o = \frac{1}{T_1} + \frac{1}{T_2}$, $T_g = \frac{1}{T_1 T_2}$ and $F_o = \frac{M_o}{T_1}$

Equation [4] is the fundamental NMR time dependent second order differential equation which can be applied to any fluid flow problem. At any given time t, we can obtain information about the system, provided that appropriate boundary conditions are applied. From this equation, we could obtain the diffusion equation, the wave equation, telephone and telegraph equations etc., and solve them in terms of NMR parameters by placing the appropriate restriction on the system as may be dictated by Eq. [4] thereby obtaining very important information about the dynamics of the system. It should be noted, however, that the term $F_o\gamma B_1(x,t)$ is the forcing function. If the function is zero, we have a freely vibrating system; else, the system is undergoing a forced vibration.

THE TIME-INDEPENDENT BLOCH NMR FLOW EQUATION

For a steady flow, all partial derivatives with respect to time can be set to zero (time independent). Hence, Eqs. [1]–[3] become (*75*, *76*, *77*):

$$v^2\frac{d^2 M_y}{dx^2} + vT_o\frac{dM_y}{dx} + \left(T_g + \gamma^2 B_1^2(x)\right)M_y = \frac{M_o}{T_1}\gamma B_1(x) \quad [5]$$

Equation [5] is a time independent Bloch NMR flow equation.

THE TIME-DEPENDENT BLOCH NMR FLOW EQUATION

For a flow that is independent of the space coordinate, x; that is, the magnetization does not change appreciably over a large x for a very long time, all partial derivatives with respect to x could be set to zero (time dependent). Hence, Eqs. [1]–[3] become (*78*):

$$\frac{d^2 M_y}{dt^2} + T_o\frac{dM_y}{dt} + \left(T_g + \gamma^2 B_1^2(t)\right)M_y = \frac{M_o}{T_1}\gamma B_1(t) \quad [6]$$

Equation [6] is a time dependent Bloch NMR flow equation which can be solved for appropriate physical situations. Detailed step by step derivations of Eq. [6] has been given in a separate study (78).

DIFFUSION IN NUCLEAR MAGNETIC RESONANCE

Generally, fluids have random molecular motions of spins, which contribute to signal loss or signal attenuation. As the NMR sensitive molecules move they carry their magnetic moments with them. Because they move very short distances, they diffuse over a small distance. This leads to an irreversible blurring of the magnetization grating and hence a loss in high spatial frequency information. That is, molecule diffusion is embedded in the NMR signal, provided we can properly interpret the information. The influence of the spin's motion on the magnetization grating is included in the Bloch equation by adding a diffusion term. This is exactly what Torrey did. The Bloch-Torrey equation is a generalization of the Bloch equations, which includes added terms due to the transfer of magnetization by diffusion (*15*).

Another form of the Bloch equation is given as:

$$\frac{d\vec{M}}{dt} = \gamma \cdot \left(\vec{M} \times \vec{B}_0\right)_{\text{excitation in B}_0} + \begin{pmatrix} \frac{M_x}{T_2} \\ \frac{M_y}{T_2} \\ \frac{M_0 - M_z}{T_1} \end{pmatrix}_{\text{transverse, longitudinal relaxation}} \quad [7]$$

The Bloch–Torrey equation is then obtained by adding the diffusion term to Eq. [7] as follows

$$\frac{d\vec{M}}{dt} = \gamma \cdot \left(\vec{M} \times \vec{B}_0\right)_{\text{excitation in B}_0} + \begin{pmatrix} \frac{M_x}{T_2} \\ \frac{M_y}{T_2} \\ \frac{M_0 - M_z}{T_1} \end{pmatrix}_{\text{transverse, longitudinal relaxation}} + D\nabla^2\vec{M}_{diffusion} \quad [8]$$

The implication of the Bloch–Torrey equation is that we could obtain a signal when there is no diffusion process. Although this is quite possible under isolated cases, however, within the realm of NMR medical imaging, it is quite impossible to rule out diffusion process under any circumstance. This means that all the magnetic resonance signals that we observe in any medical imaging or spectroscopic methods are a consequence of all possible processes (gradient, chemical shift, susceptibility, couplings) taking place within the body under investigation, including diffusion. Therefore, it would be very important and more fundamental to derive the diffusion system directly from the Bloch NMR flow equations. In doing so, we shall take a look at Eq. [4], based on some set of mathematical assumptions.

MATHEMATICAL ASSUMPTIONS AND JUSTIFICATION

Before we derive the differential equations that are fundamental for the analyses of NMR/MRI experiments, it would be necessary to state the justifications for two very important assumptions that is needed be made in this presentation. The assumptions are as follows:

$$\frac{\partial^2 M_y}{\partial t^2} + 2v\frac{\partial^2 M_y}{\partial x \partial t} + vT_o\frac{\partial M_y}{\partial x} + \left\{T_g + \gamma^2 B_1^2(x,t)\right\}M_y = 0 \quad [9]$$

It is worthy of note that the term $T_g + \gamma^2 B_1^2(x,t)$ is very important in all magnetic resonance systems. In any given NMR system, we may write $\Omega = \frac{1}{T_1T_2} + \gamma^2 B_1^2(x,t)$; in which case, the following possibilities may be encountered.

i. $\Omega = \frac{1}{T_1T_2} + \gamma^2 B_1^2$, where B_1 is constant–independent of x and t.

ii. $\Omega = \frac{1}{T_1T_2} + \gamma^2 B_1^2(x)$, where $B_1(x)$ is independent of t.

iii. $\Omega = \frac{1}{T_1T_2} + \gamma^2 B_1^2(t)$, where $B_1(t)$ is independent of x.

iv. $\Omega \approx \frac{1}{T_1T_2}$, where the condition $\frac{1}{T_1T_2} \gg \gamma^2 B_1^2(x,t)$ holds.

v. $\Omega \approx \gamma^2 B_1^2$, where $\frac{1}{T_1T_2} \ll \gamma^2 B_1^2$. and B_1 is constant–independent of x and t

vi. $\Omega \approx \gamma^2 B_1^2(x,t)$, where $\frac{1}{T_1T_2} \ll \gamma^2 B_1^2(x,t)$ holds

vii. $\Omega \approx \gamma^2 B_1^2(x)$, where $\frac{1}{T_1T_2} \ll \gamma^2 B_1^2(x)$ and B_1 is independent t

viii. $\Omega \approx \gamma^2 B_1^2(t)l$, where $\frac{1}{T_1T_2} \ll \gamma^2 B_1^2(t)$ and B_1 is independent of x.

In multiple dimensions (three dimensions mostly), the radio frequency field has the form $\gamma B_1(x,t) \equiv \gamma B_1(\vec{r},t)$. If the NMR system is designed in such a way that the transverse magnetization takes the form of a plane wave:

$$M_y = Ae^{mx+nt} \quad [10]$$

where m and n are dependent on the NMR parameters. If n takes the value

i. $n = -vm + (v^2m^2 - \Omega)^{1/2}$ or $n = -vm - (v^2m^2 - \Omega)^{1/2}$, Eq. [10] holds

ii. $n = -vm + (v^2m^2 - vT_0m - \Omega)^{1/2}$ or $n = -vm - (v^2m^2 - vT_0m - \Omega)^{1/2}$ Eq. [10] holds; and once the value of either n or m is fixed, the other follows directly.

However, the form of NMR signal shown in Eq. [10] is only possible if Ω is constant. That is, Eq. [10] can be a solution to Eq. [9], only if the coefficients of the differential equation are independent of both x and t. Therefore, cases (i), (iv), and (v) provide the necessary justification for the assumptions made in Eqs. [9] and [10].

As an example, if we set $v^2m^2 - \Omega = 0$ in condition (i), then, $m = \frac{\sqrt{\Omega}}{v}$ and $n = -\sqrt{\Omega}$. The transverse magnetization takes the form:

$$M_y = Ae^{\frac{\sqrt{\Omega}}{v}x + \sqrt{\Omega}t} \qquad [11]$$

Similarly, if we set $m = \frac{\sqrt{\Omega}}{v}$ in case (i), then $n = -\sqrt{\Omega} \pm i\left(T_0\sqrt{\Omega}\right)^{1/2}$ and the transverse magnetization takes the form:

$$M_y = Ae^{\frac{\sqrt{\Omega}}{v}x - \sqrt{\Omega}t + i\left(T_0\sqrt{\Omega}\right)^{1/2}t} + Ae^{\frac{\sqrt{\Omega}}{v}x - \sqrt{\Omega}t - i\left(T_0\sqrt{\Omega}\right)^{1/2}t} \qquad [12]$$

or if, $m = -\frac{\sqrt{\Omega}}{v}$ in case (i), then, $n = \sqrt{\Omega} \pm \left(T_0\sqrt{\Omega}\right)^{1/2}$, the transverse magnetization becomes

$$M_y = Ae^{\frac{\sqrt{\Omega}}{v}x + \sqrt{\Omega}t + \left(T_0\sqrt{\Omega}\right)^{1/2}t} + Ae^{\frac{\sqrt{\Omega}}{v}x + \sqrt{\Omega}t - \left(T_0\sqrt{\Omega}\right)^{1/2}t} \qquad [13]$$

Provided the assumption in Eq. [9] holds, we could therefore write:

$$\frac{\partial M_y}{\partial t} = \frac{-v^2}{T_o}\frac{\partial^2 M_y}{\partial x^2} + \frac{F_o}{T_o}\gamma B_1(x,t) \qquad [14]$$

Similarly, if we set:

$$D = \frac{-v^2}{T_o} \qquad [15]$$

$$\frac{\partial M_y}{\partial t} = D\frac{\partial^2 M_y}{\partial x^2} + \frac{F_o}{T_o}\gamma B_1(x,t) \qquad [16]$$

If parameter D represents the diffusion coefficient, then Eq. [16] is the equation of diffusion of magnetization as the nuclear spins move. The function $\frac{F_o}{T_o}\gamma B_1(x,t)$ is the forcing function, which shows that the application of rf B_1 field has an influence on the diffusion of magnetization within a voxel. It is interesting to note that the dimension of Eq. [16] exactly matches that of diffusion coefficient. Equation [16] as would be observed, is only applicable when D in nondirectional. That is, we have a constant diffusion coefficient. The model would work quite well for molecules that move very short distances over a very considerable amount of time.

DIFFUSION IN MULTIDIMENSIONS WITH CONSTANT DIFFUSION COEFFICIENT

Even with a constant diffusion coefficient, we may observe that the transverse magnetization varies with more than one space coordinates. In such a case, we may generalize Eq. [16] as follows (within some limited mathematical error, if any):

$$\frac{\partial M_y}{\partial t} = D\nabla^2 M_y + \frac{F_o}{T_o}\gamma B_1(\vec{r},t) \qquad [17]$$

where $\bigtriangledown^2$ is the Laplacian in the coordinate system that correctly describe the slice under investigation (Cartesian, cylindrical or spherical geometries). Equation [17] is the NMR diffusion equation.

DIFFUSION WITH VARIABLE DIFFUSION COEFFICIENT

In any situation in which the diffusion coefficient varies in different directions, it is very crucial to quantify them as they are because they hold important information about the system and what is happening to the system at any instant. Hence, we may write Eq. [16] as

$$\frac{\partial M_y}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial M_y}{\partial x}\right) + \frac{F_o}{T_o}\gamma B_1(\vec{r},t) \qquad [18]$$

Equation [18] is known as the one-dimensional NMR diffusion equation with variable coefficient (*61*, *64*, *66*, *67*). It is worthy of note that the assumptions that led us to Eq. [16] may not fully apply here because of the spatial dependence of the diffusion coefficient. However, it is common place in applied mathematics that constant coefficient differential equations have variable coefficient analogs associated with them (80, 81, 82).

DIFFUSION IN MULTIDIMENSIONS WITH VARIABLE DIFFUSION COEFFICIENT

Variable coefficient diffusion processes are more pronounced in multi-dimensional problems. Because magnetic resonance processes are generally multi-dimensional if we must pay attention to very small changes, it would be of utmost importance to discuss how the NMR diffusion coefficient is expected to change in different directions. Therefore, we shall generalize Eq. [18] as follows

$$\frac{\partial M_y}{\partial t} = \nabla \cdot \left(D \nabla M_y\right) + \frac{F_o}{T_o}\gamma B_1(\vec{r}, t) \qquad [19]$$

where $\bigtriangledown$ is the Del operator in the coordinate system that correctly describe the slice under investigation (Cartesian, cylindrical, or spherical).

ADVECTION–DIFFUSION IN NUCLEAR MAGNETIC RESONANCE

In the earlier sections, we discussed the diffusion of transverse magnetization M_y in a fluid which is approximately static within the slice under investigation. In such a slice, the signal attenuation observed is not due to nuclear spin moving away from the imaging slice being investigated; but due to small molecular motion within the region and molecular interaction.

However, to investigate the diffusion process of magnetization in a fluid moving at a uniform velocity, v, which is constant in time, we have to take the process of advection into consideration. The equation which describes such a process is known as the advection equation. The advection equation (*80*, *81*) is the partial differential equation that governs the motion of a conserved scalar as it is advected by a known velocity field. It is derived using the scalar's conservation law, together with Gauss's theorem, and taking the infinitesimal limit. The diffusion–advection equation (a differential equation describing the process of diffusion and advection) is obtained by adding the advection operator to the main diffusion equation. In the Cartesian coordinates, the advection operator (*80*, *81*) is

$$\vec{v} \cdot \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

where the velocity vector $\boldsymbol{v}$ has components v_x, v_y, and v_z in the x, y, and z directions, respectively.

Based on the assumptions in Eqs. [10] and provided that $n = -vm \pm (v^2 m^2 - \Omega)^{1/2}$ (where Ω is independent of x and t), we could write

$$\frac{\partial^2 M_y}{\partial t^2} + 2v\frac{\partial^2 M_y}{\partial x \partial t} + \left\{T_g + \gamma^2 B_1^2(x,t)\right\} M_y = 0 \qquad [20]$$

It follows that

$$vT_o \frac{\partial M_y}{\partial x} + T_o \frac{\partial M_y}{\partial t} = -v^2 \frac{\partial^2 M_y}{\partial x^2} + F_o \gamma B_1(x,t) \qquad [21]$$

$$v\frac{\partial M_y}{\partial x} + \frac{\partial M_y}{\partial t} = -\frac{v^2}{T_o}\frac{\partial^2 M_y}{\partial x^2} + \frac{F_o}{T_o}\gamma B_1(x,t) \qquad [22]$$

provided that

$$D = -\frac{v^2}{T_o} \qquad [23]$$

and

$$v\frac{\partial M_y}{\partial x} + \frac{\partial M_y}{\partial t} = D\frac{\partial^2 M_y}{\partial x^2} + \frac{F_o}{T_o}\gamma B_1(x,t) \qquad [24]$$

where D is the diffusion coefficient, and because v is the fluid velocity, Eq. [24] is the diffusion–advection equation for the NMR magnetization. It is very interesting to note that Eq. [24] exactly matches the advection equation without any special transformation whatsoever. Equation [24] is exactly the one-dimensional NMR diffusion–advection equation with constant coefficient. This means that the fluid velocity v and the diffusion coefficient D are constant in x.

ONE-DIMENSIONAL DIFFUSION–ADVECTION WITH VARIABLE DIFFUSION COEFFICIENT

If the fluid velocity and the diffusion coefficient D are both dependent on the space variable x, we need to re-write Eq. [24] as follows:

$$\frac{\partial}{\partial x}\left(vM_y\right) + \frac{\partial M_y}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial M_y}{\partial x}\right) + \frac{F_o}{T_o}\gamma B_1(x,t) \qquad [25]$$

In this case, we may write $v = v(x)$ and $D = D(x)$.

DIFFUSION–ADVECTION IN MULTIDIMENSIONS WITH CONSTANT DIFFUSION COEFFICIENT

When the transverse magnetization could be observed to change considerably in different spatial directions, we may generalize Eq. [24] as (within some limited mathematical error, if any):

$$v\nabla M_y + \frac{\partial M_y}{\partial t} = D\nabla^2 M_y + \frac{F_o}{T_o}\gamma B_1(\vec{r},t) \qquad [26]$$

where $\bigtriangledown^2$ is the Laplacian and $\bigtriangledown$ is the Del operator ($\vec{v}\cdot\nabla$ is the advection operator), in the coordinate system that correctly describes the slice under investigation. D and v are both independent of the spatial coordinate x. We shall write out Eq. [26] in different coordinate system because of their often confusing nature:

Rectangular Geometries: The NMR diffusion–advection equation in this case has the form

$$v\left(\frac{\partial M_y}{\partial x}+\frac{\partial M_y}{\partial y}+\frac{\partial M_y}{\partial z}\right)+\frac{\partial M_y}{\partial t} = D\left(\frac{\partial^2 M_y}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+\frac{\partial^2 M_y}{\partial z^2}\right)+\frac{F_o}{T_o}\gamma B_1(\vec{r},t) \quad [27]$$

Cylindrical Geometries: In cylindrical geometry, the NMR diffusion - advection equation is given as

$$v\left(\frac{\partial M_y}{\partial r}+\frac{1}{r}\frac{\partial M_y}{\partial \phi}+\frac{\partial M_y}{\partial z}\right)+\frac{\partial M_y}{\partial t} = D\left(\frac{\partial^2 M_y}{\partial r^2}+\frac{1}{r}\frac{\partial M_y}{\partial r}+\frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2}+\frac{\partial^2 M_y}{\partial z^2}\right)+\frac{F_o}{T_o}\gamma B_1(\vec{r},t) \quad [28]$$

Spherical Geometry: The NMR diffusion–advection equation in spherical geometries has the form

$$v\left(\frac{\partial M_y}{\partial r}+\frac{1}{r}\frac{\partial M_y}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial M_y}{\partial \phi}\right)+\frac{\partial M_y}{\partial t} = D\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial M_y}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 M_y}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial M_y}{\partial \theta}\right)\right)+\frac{F_o}{T_o}\gamma B_1(\vec{r},t) \quad [29]$$

DIFFUSION–ADVECTION IN MULTIDIMENSIONS WITH VARIABLE DIFFUSION COEFFICIENT

The NMR diffusion–advection equation with variable coefficient could be obtained from Eq. [26] as (*69*):

$$\nabla\cdot\left(vM_y\right)+\frac{\partial M_y}{\partial t} = \nabla\cdot\left(D\nabla M_y\right)+\frac{F_o}{T_o}\gamma B_1(\vec{r},t) \quad [30]$$

We have to re-write the advection operator (the first term on the left hand side of Eq. [30]) because the fluid velocity is now spatially dependent.

Generally speaking, the advection term for the transverse magnetization is $\nabla\cdot\left(vM_y\right)$, the expansion of which is given as:

$$\nabla\cdot\left(vM_y\right) = (\nabla\cdot v)M_y + v\cdot\nabla M_y$$

When the fluid velocity is constant, $\nabla.v = 0$ and then, $\nabla\cdot\left(vM_y\right) = v\nabla M_y$. This is very similar to the case of incompressible fluid in fluid dynamics (*83*).

Rectangular Geometries: The NMR diffusion equation describing this case has the form (*71*, *72*)

$$\frac{\partial\left(v_x M_y\right)}{\partial x}+\frac{\partial\left(v_y M_y\right)}{\partial y}+\frac{\partial\left(v_z M_y\right)}{\partial z}+\frac{\partial M_y}{\partial t} = \frac{\partial}{\partial x}\left(D_x\frac{\partial M_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y\frac{\partial M_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(D_z\frac{\partial M_y}{\partial z}\right)+\frac{F_o}{T_o}\gamma B_1(\vec{r},t) \quad [31]$$

Cylindrical Geometries: If the slice or voxel has a cylindrical geometry, the NMR diffusion equation gives (*71*, *72*)

$$\frac{\partial\left(v_r M_y\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(v_\phi M_y\right)}{\partial \phi}+\frac{\partial\left(v_z M_y\right)}{\partial z}+\frac{\partial M_y}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(D_r r\frac{\partial M_y}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(D_\phi\frac{\partial M_y}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(D_z\frac{\partial M_y}{\partial z}\right)+\frac{F_o}{T_o}\gamma B_1(\vec{r},t) \quad [32]$$

We have used the definition for the advection term as given by the term $\nabla\cdot\left(vM_y\right)$.

Spherical Geometries: For a slice or voxel that has spherical geometry, the equation becomes (*71*, *72*):

$$\frac{\partial}{\partial r}\left(v_r M_y\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(v_\theta M_y\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(v_\phi M_y\right)+\frac{\partial M_y}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(D_r r^2\frac{\partial M_y}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(D_\theta\sin\theta\frac{\partial M_y}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(D_\phi\frac{\partial M_y}{\partial \phi}\right)+\frac{F_o}{T_o}\gamma B_1(\vec{r},t) \quad [33]$$

A MODEL APPLICATION OF THE MATHEMATICAL FORMULATION FOR THE ANALYSIS OF DIFFUSION MAGNETIC RESONANCE ANGIOGRAPHY

About 55% of the blood is composed of a liquid known as plasma. The rest of the blood is made of

three major types of cells: red blood cells (also known as erythrocytes), white blood cells (leukocytes), and platelets (thrombocytes). The blood plasma consists predominantly of water and salts-which makes them very good for NMR analysis. The flow of blood within its vessels is made up of two very obvious diffusion processes which in actual sense, consequence of a single physiological process; the diffusion of blood components from one point in the vessel to the other and the diffusion of same across the vessel walls. Although they are well connected, only the latter is a normal diffusion process. However, for proper monitoring of the delivery of blood components with reference to the flow within the vessels, we need to use the former in which the components diffuse in the forward direction up to the point at which they are absorbed across the walls. Provided that the blood component under investigation is NMR sensitive, we could follow its diffusion by monitoring its NMR signal emitted at its unique resonant frequency such that immediately the substance is absorbed into the interstitial, the NMR signal decays. As long as they are not all absorbed at the same point, a successful absorption is observed as decaying signal along the direction of blood flow. This view can be very important in the study of sickle cell anemia and finding a treatment to the genetic disorder. This is because, we may be able to visualize blood vessels pores that are blocked by the sickle cell and find out the factor that makes the erythrocytes rigid. It is expected that the diamagnetic property of the oxyhemoglobin that makes it difficult to deliver the oxygen on it would be very useful here. Hence, a very slowly decaying signal or nondecaying signal could be the indication of some form of problem with the vessel.

Therefore, a very porous blood vessel (normal blood vessel) would appear as if it is a pipe that is allowing free diffusion of the components with a substantial diffusion coefficient while a problematic vessel would either have its dimension reduced (when stenosis is present) or it would appear to be reduced in dimension (especially when the pores are blocked such that diffusing components are unable to enter the interstitial fluid and so they are always hanging around while new ones from the arterial blood are showing up). Then we may assume that the diffusion coefficient is a function of the location so that we can characterize it from point to point.

BLOOD FLOW AND DIFFUSION IN VESSELS WITH CONSTANT DIMENSIONS

It is expected that in normal blood flow, the motion is slow well enough such that components diffuse almost freely and the normal blood pressure in this circumstance makes the vessels dimensions approximately constant while blood flows through and then we say that the diffusion coefficient is fairly constant. We shall consider the blood vessel as a cylindrical structure whose radius does not change substantially (there are limited dilation and constriction).Therefore, we have a diffusion process given by Eq. [17] as (*71*, *72*)

$$\frac{\partial M_y}{\partial t} = D\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M_y}{\partial \phi^2} + \frac{\partial^2 M_y}{\partial z^2}\right) + \frac{F_o}{T_o}\gamma B_1(\vec{r}, t) \quad [34]$$

Provided that the RF power received by each blood component within the voxel is independent of their spatial location and because the transverse magnetization would only vary in the direction of fluid flow (taken to be the z-direction) and in the direction of component absorption (the radial direction) into the vessel walls, the diffusion equation becomes

$$\frac{\partial M_y}{\partial t} = D\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{\partial^2 M_y}{\partial z^2}\right) + \frac{F_o}{T_o}\gamma B_1(t) \quad [35]$$

If we write $M_y(r, z, t) = M_{yc}(r, z, t) + w_c(t)$

$$w_c' = \frac{F_o}{T_o}\gamma B_1(t) \quad [36]$$

It follows that $\frac{\partial M_{yc}}{\partial t} = D\left(\frac{\partial^2 M_{yc}}{\partial r^2} + \frac{1}{r}\frac{\partial M_{yc}}{\partial r} + \frac{\partial^2 M_{yc}}{\partial z^2}\right)$

By the usual separation of variables method, we have the following two equations

$$\frac{dG_{yc}(t)}{dt} = -\alpha_0^2 D G_{yc}(t) \quad [37]$$

$$\frac{1}{R_{yc}}\frac{d^2 R_{yc}}{dr^2} + \frac{1}{rR_{yc}}\frac{dR_{yc}}{dr} = -\frac{1}{Z_{yc}}\frac{d^2 Z_{yc}}{dz^2} - \alpha_0^2 \quad [38]$$

Equation [38] must also be equal to a constant ξ_0^2

$$\frac{1}{R_{yc}}\frac{d^2 R_{yc}}{dr^2} + \frac{1}{rR_{yc}}\frac{dR_{yc}}{dr} = -\frac{1}{Z_{yc}}\frac{d^2 Z_{yc}}{dz^2} - \alpha_0^2 = -\xi_0^2$$

giving two distinct differential equations (*71*, *72*):

$$\frac{d^2 R_{yc}}{dr^2} + \frac{1}{r}\frac{dR_{yc}}{dr} + \xi_0^2 R_{yc} = 0 \quad [39]$$

$$\frac{d^2 Z_{yc}}{dz^2} = -\left(\alpha_0^2 - \xi_0^2\right) Z_{yc} \quad [40]$$

Equation [37] has the solution:

$$G_{yc}(t) = a_{21}e^{-\alpha_0^2 Dt} \quad [41]$$

Equation [39] is the Bessel differential equation (of order zero), the solution of which is given as

$$R_{yc}(r) = a_{22}J_0(\xi_0 r) + a_{23}Y_0(\xi_0 r) \quad [42]$$

We shall require that our solution be finite at the origin so that Bessel function of the second kind is equated to zero

$$R_{yc}(r) = a_{22}J_0(\xi_0 r) \quad [43]$$

Also, the solution to Eq. [40] gives that

$$Z_{yc}(z) = a_{24}\cos\left(\sqrt{\alpha_0^2 - \xi_0^2}\right)z + a_{25}\sin\left(\sqrt{\alpha_0^2 - \xi_0^2}\right)z \quad [44]$$

$$M_{yc} = J_0(\xi_0 r)\left\{A_{24}\cos\left(\sqrt{\alpha_0^2 - \xi_0^2}\right)z + B_{24}\sin\left(\sqrt{\alpha_0^2 - \xi_0^2}\right)z\right\}e^{-\alpha_0^2 Dt} \quad [45]$$

From Eqs. [34] and [35], we have

$$M_y = J_0(\xi_0 r)\left\{A_{24}\cos\left(\sqrt{\alpha_0^2 - \xi_0^2}\right)z + B_{24}\sin\left(\sqrt{\alpha_0^2 - \xi_0^2}\right)z\right\}e^{-\alpha_0^2 Dt} + \int_0^{t_o}\frac{F_o}{T_o}\gamma B_1(t)dt \quad [46]$$

where t_o is defined as usual. At maximum signal, the equilibrium magnetization factor, F_o in Eq. [46] is equal to zero, if we put:

$$\alpha_0^2 = \xi_0^2 = \gamma^2 G^2 \delta^2$$

Equation [46] becomes a Gaussian distribution of the form:

$$\ln\left(\frac{M_y}{A_{24}J_0(\xi_0 r)}\right) = -\xi_0^2 Dt \quad [47]$$

The simple expression in Eq. [47] is known as the Stejskal-Tanner formula for diffusive attenuation which highlights how variations in the spatial scale parameter, ξ_0 and the diffusion time t, independently affect the signal attenuation. In a medium in which water does not move freely—in which it is confined within internal compartments (restricted), is reflected by obstacles but still free to meander relatively freely throughout the medium (hindered), or experiences higher diffusivity in some directions and lower in others (anisotropic)—this simple relationship no longer holds. Still, the approach used to derive Eq. [47] can be generalized to treat each of these more complex cases. Therefore, the imaging equation takes the form (*78*, *81*):

$$S(t) = \int J_0(\lambda t)e^{-Dt}dt = \frac{1}{\sqrt{\lambda^2 + D^2}} = \frac{1}{\sqrt{(\xi_0 v)^2 + D^2}} \quad [48]$$

where $r = vt$ and $\xi_0 v = \lambda$

Figures 1 and 2 demonstrate the variations of the output signal S(t) with the parameters $\lambda = \xi_0 v$ and the diffusion coefficient, D, based on Eq. [6]. The different choices of velocities and diffusion coefficient are indicative of specific physiological processes (for example, blood vessels are of different sizes, diffusion coefficients and hence the varying speed of blood flows) so that for any combination of D and v, we have a knowledge of the magnetic resonance setup for which serious signal amplification is required. It is shown that the fluid velocity v, diffusion coefficient D and the NMR imaging parameter ξ_0 can easily be inferred from the NMR signal. However, D does not have significant effect on the values of $S(t)$ because we considered blood vessel of uniform cross section. It is observed in Fig. 1 that the NMR signal decreases significantly with increase in fluid velocity. This indicates that the analytical method applied in this study can be very useful at the molecular level. In addition, variation of the NMR signal with the NMR parameter ξ_0 can be very useful in magnetic resonance angiography as a good complement to the Stejskal–Tanner formulation.

DISCUSSIONS

Bloch NMR flow equations as discussed in detail above are not just theoretical exercises but can be used to characterize and solve real life problems in an interdisciplinary and multidisciplinary way. The basic equations derived from the Bloch NMR flow equations are quite interesting, motivating and exciting. The NMR diffusion differential equations presented in various geometries and coordinates is an

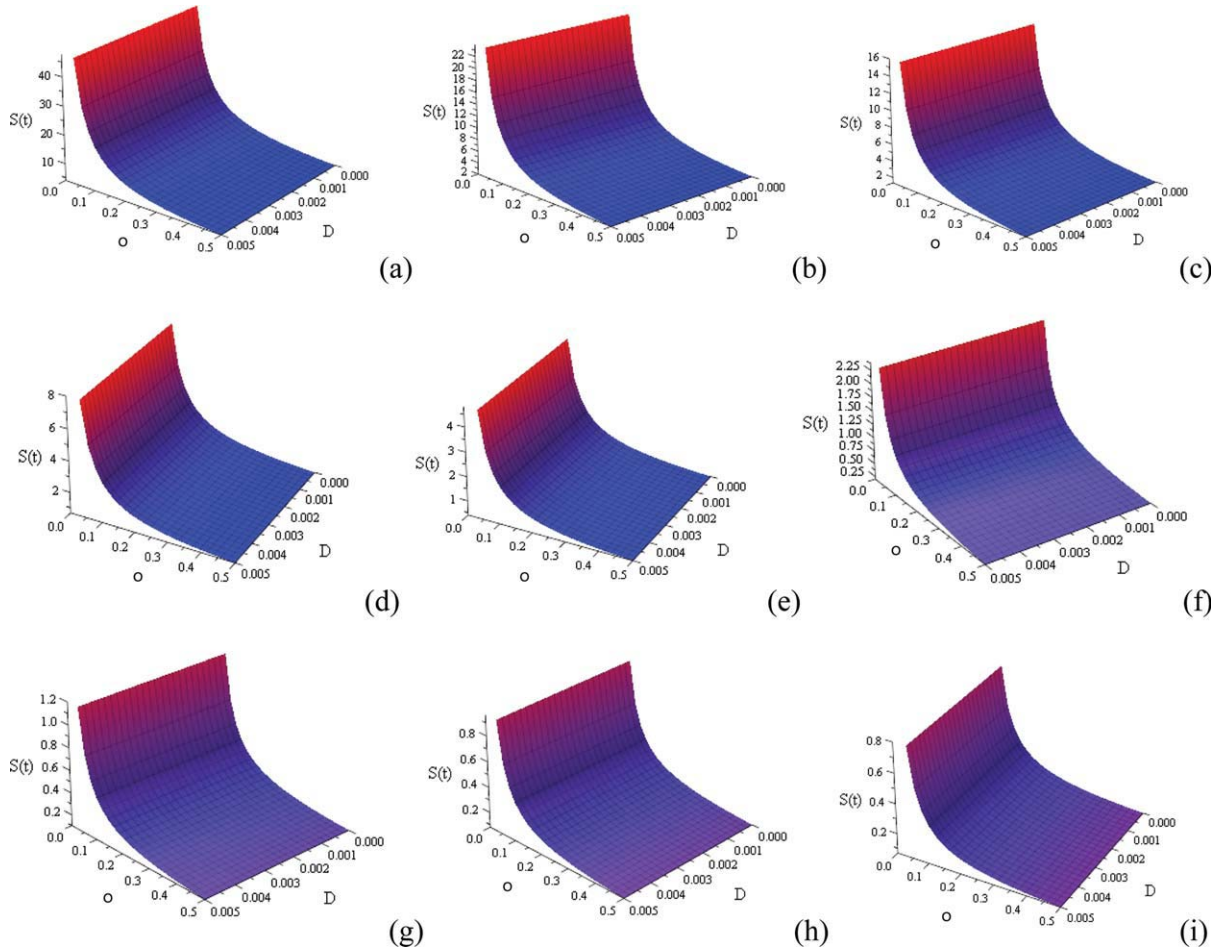

Figure 1 Plots of the output signal $S(t)$ against the diffusion coefficient D and the parameter ξ_o for fluid velocities v (a) 0.5 m/s, (b) 1.0 m/s, (c) 1.5 m/s, (d) 3.0 m/s, (e) 5.0 m/s, (f) 10.0 m/s, (g) 20.0 m/s, (h) 25.0 m/s, (i) 30 m/s.

intrinsic property of the fundamental Bloch NMR flow equations which can be solved to accurately extract the MRI physics direct from the Bloch NMR equations without the need to import the diffusion term. These equations have been solved analytically in the specified coordinates for the analyses of diffu-

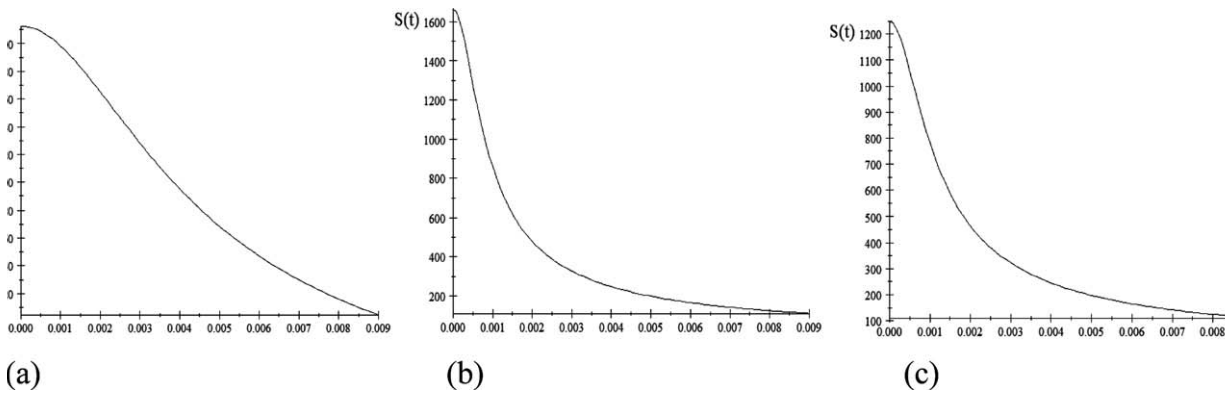

Figure 2 Plot of the output signal $S(t)$ against $\lambda = \xi_0 v$ for (a) cerebrospinal fluid ($D = 3.2 \times 10^3 m^2/s$) (b) white matter of cerebrum ($D = 6 \times 10^4$ m^2/s) (c) grey matter of cerebrum ($D = 8 \times 10^4$ m^2/s).

sion and perfusion processes in restricted geometries (*66–79*).

It can be interesting to note that Eq. [10] gives the general form of the (complex) transverse magnetization of the Bloch NMR flow equation. Equation [10] is the magnetic resonance imaging equation (*75*) which can be controlled depending on the values assigned to the constants m and n. If $m = (i\gamma g t\ \ k) > n$, the imaging equation then takes the form

$$S(t) = A \int M_o(x) e^{-i\gamma g t.x} dx$$

where $M_o(x) = e^{kx}$ and, $A = e^{k\Phi}$ is a complex arbitrary constant. A good choice for a collection of pulse sequences has been shown (*84*).

Although, a translational mechanical analysis of Eq. [6] has been presented (*78*), it may be informative to mention that the response of certain network of tuned circuits to a unit NMR impulse applied in the first circuit (measurement) can be described by the set of Eq. [6] where $\Omega = 1$ as given below:

$$\left.\begin{array}{l} \frac{d^2M_{y1}}{dt^2} + T_o\frac{dM_{y1}}{dt} + M_{y1} = \frac{M_o}{T_1}\gamma B_1(t) = I(t) \\ \frac{d^2M_{y2}}{dt^2} + T_o\frac{dM_{y2}}{dt} + M_{y2} = M_{y1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d^2M_{y\eta}}{dt^2} + T_o\frac{dM_{y\eta}}{dt} + M_{y\eta} = M_{y\eta-1} \end{array}\right\} \quad [49]$$

The transverse magnetization, $M_{y\eta}$ is thus given as follows:

$$M_{y\eta} = \frac{\Gamma[(2\eta+1)/2]}{\Gamma(2\eta)}\left(\frac{2t}{\rho}\right)^{(2\eta+1)/2} e^{\frac{T_o}{2}t} J_{(2\eta+1)/2}(\rho t) \quad [50]$$

where, $\rho = 1 - \frac{T_o^2}{4}$, J(ρt) is Bessel function of order η with a parameter ρ and Γ( ) is the gamma function. Then,

$$M_{y\eta} = \frac{\Gamma[(2\eta+1)/2]}{\Gamma(2\eta)}\left(\frac{8t}{4-T_o^2}\right)^{(2\eta+1)/2} e^{\frac{T_o}{2}t} J_{(2\eta+1)/2}\left((1-\frac{T_o^2}{4})t\right) \quad [51]$$

The plots in Figs. 3 and 4 show the interesting interplay of the time, relaxation rate the transverse magnetization in translational mechanical analysis. As shown above, the transverse magnetization peaks up on both sides of the time axis at specific values of the relaxation rate T_o (Eq. [51]) and Figs. 3 and 4 demonstrate the importance of Eqs. [49], [50], and [51] in spectroscopic studies where the relaxation times provide the major magnetic resonance contrast. Different molecules resonate at different frequencies and this has direct bearing on the relaxation rate T_o, which gives different values for different molecules and molecular environment. The general behavior of the magnetic resonance signals (in terms of the transverse magnetization M_y) as T_o and times varies are demonstrated in the figures. Slight variations are definitely expected for different molecules and similar molecules taking part in different chemical reactions using the analytical solutions presented in this study but such variations would always present a functional behavior as shown in the figures.

It should be mentioned that using appropriate mathematical procedures, Eq. [4] can also be transformed into wave-like equation

$$v^2\frac{\partial^2 M_y}{\partial x^2} + \frac{\partial^2 M_y}{\partial t^2} = \frac{M_o}{T_1}\gamma B_1(t)$$

provided that

$$nT_o = -\frac{1}{T_1T_2},\ 2n = -T_o \quad for \quad \gamma^2B_1^2 \ll \frac{1}{T_1T_2}$$

and

$$nT_o = -\gamma^2B_1^2(t),\ 2n = -T_o\ for\ \gamma^2B_1^2(t) \gg \frac{1}{T_1T_2}$$

Similarly, Eq. [4] can be transformed into Euler's equation

$$v^2\frac{\partial^2 M_y}{\partial x^2} + 2v\frac{\partial^2 M_y}{\partial x\partial t} + \frac{\partial^2 M_y}{\partial t^2} = \frac{M_o}{T_1}\gamma B_1(t)$$

Provided that

$$2nT_o = -\frac{1}{T_1T_2}, n = vm for\ \gamma^2B_1^2 \ll \frac{1}{T_1T_2} \text{ and } 2nT_o = -\gamma^2B_1^2(t), n = vm for\ \gamma^2B_1^2(t) \gg \frac{1}{T_1T_2}$$

It is interesting to note that the wave and Euler's equations can be expressed in cylindrical and spherical coordinates to model multidimensional problems. The analytical solutions of all the differential equa-

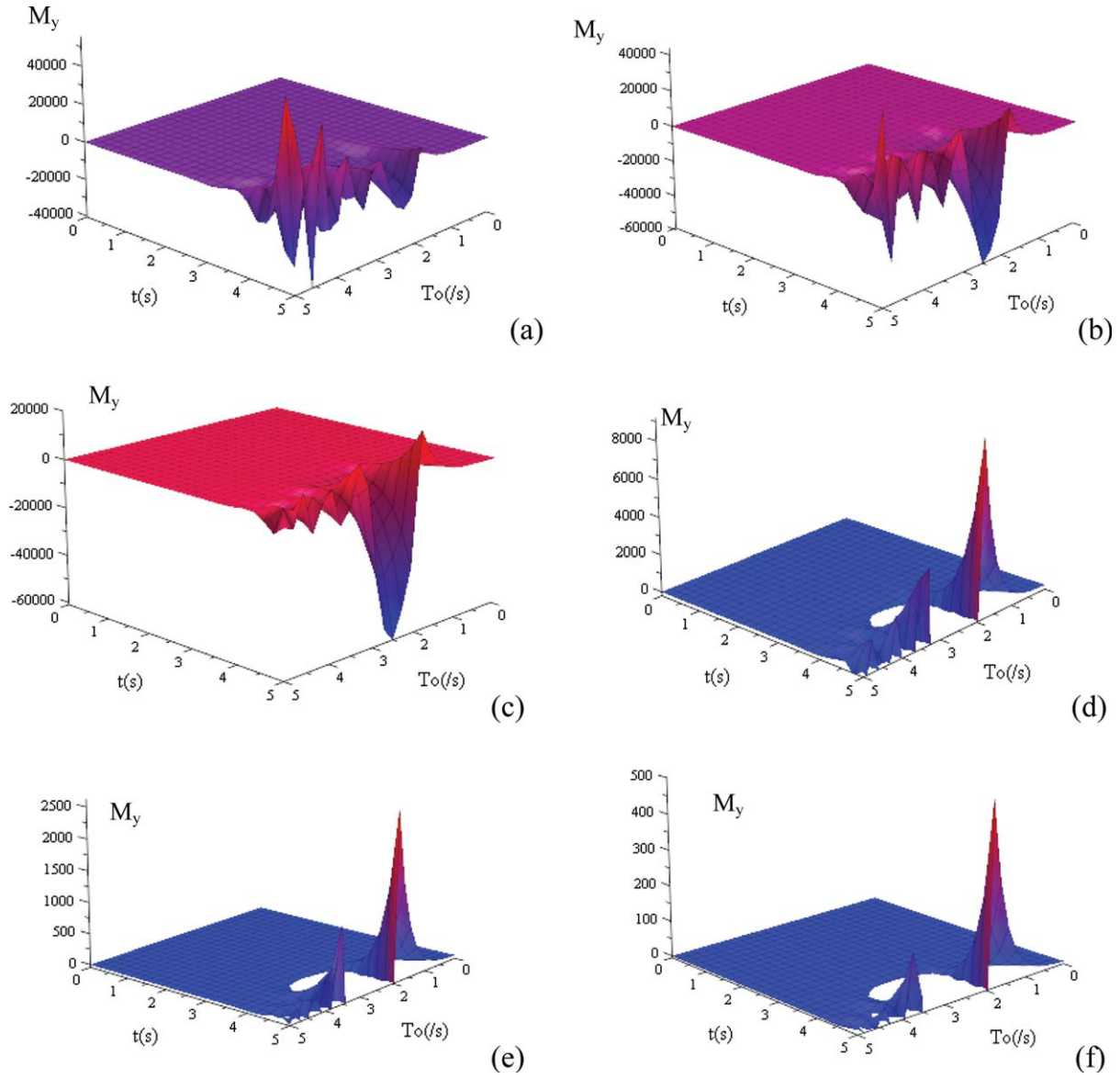

Figure 3 Plots of the transverse magnetization $M_{y\eta}$ against the relaxation rate T_o and time for (a) $\eta = 1$, (b) $\eta = 2$, (c) $\eta = 3$, (d) $\eta = 4$, (e) $\eta = 5$, (f) $\eta = 6$.

tions derived in this presentation and their NMR properties can be fundamental toward understanding the basic NMR physics of extracting the relevant flow parameters to solve fluid flow problems in physical, biological, biomedical, geophysical, environmental, communication, agricultural, and medical sciences due to the abundant mathematical tools available to solve the equations analytically. The most fascinating point about these equations and their solutions is that we may do NMR without necessarily having the big equipments and the rigorous data processing. It should be particularly noted that this study is very much concerned with analytical methods that the Bloch NMR flow equations can provide for different physical, biomedical, geophysical, medical, and environmental situations even at the molecular level for the purpose of interdisciplinary approach to solve difficult problems.

A model application of the mathematical formulation for the analysis of diffusion magnetic resonance angiography is provided as a simple illustration to show the physical interpretation of one of the eight assumptions made earlier in this presentation and the given mathematical justifications. This practical example with typical numerical and experimental parameters applied is valid

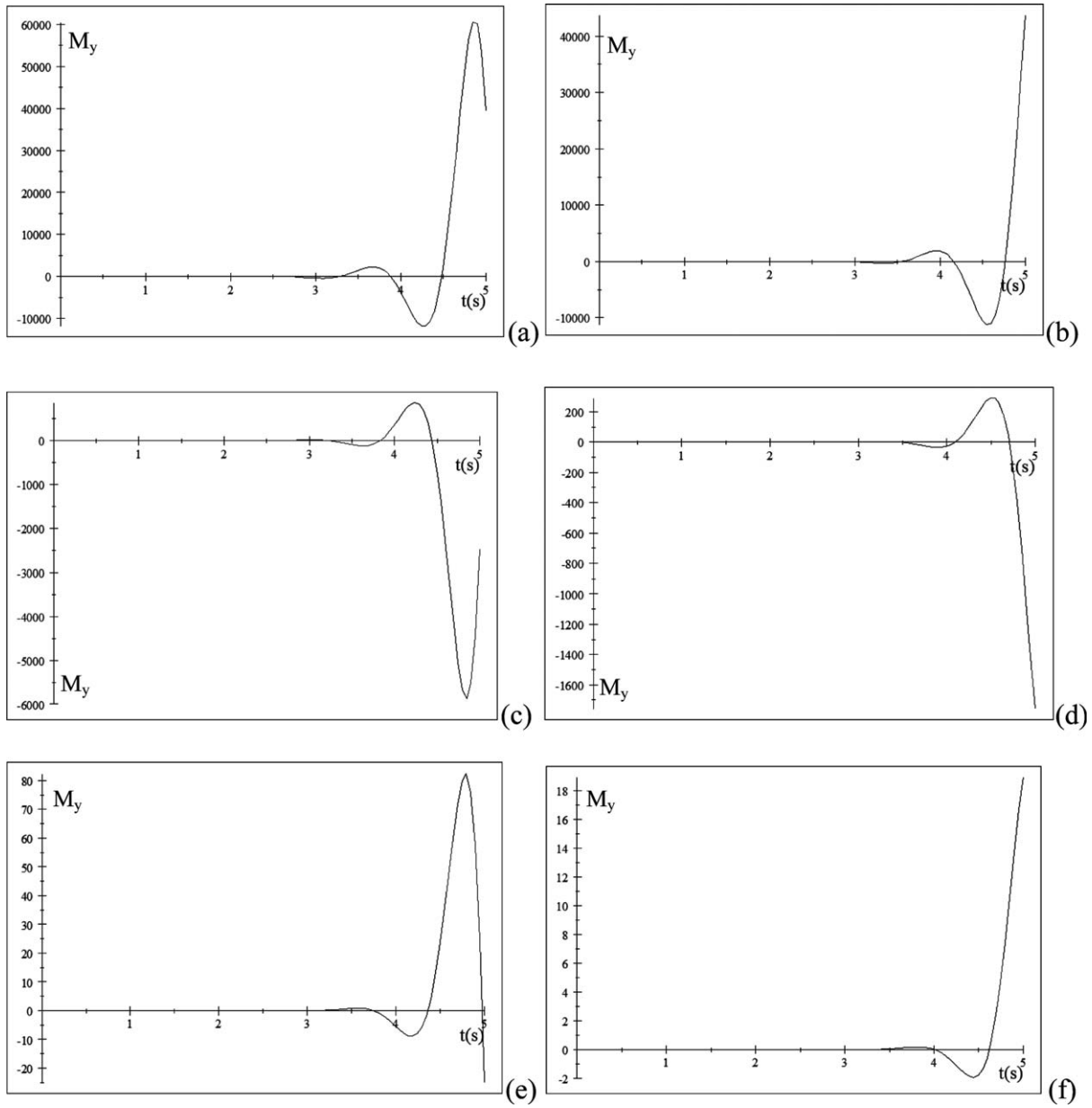

Figure 4 Plots of the transverse magnetization $M_{y\eta}$ against the relaxation the time (a) $\eta = 1$, (b) $\eta = 2$, (c) $\eta = 3$, (d) $\eta = 4$, (e) $\eta = 5$, (f) $\eta = 6$ ($T_1 = 1.00$ s, $T_2 = 0.25$ s).

only if $\Omega \approx \frac{1}{T_1 T_2}$, where the condition $\frac{1}{T_1 T_2} \gg \gamma^2 B_1^2(x,t)$ holds in Eqs. [4], [6], and [9].

Similarly, the response of certain network of tuned circuits to a unit NMR impulse applied in the first circuit (measurement) can be described by the set of Eqs. [6], [49]–[51];

where $\Omega = 1$ or $\frac{1}{T_1 T_2} = 1 - \gamma^2 B_1^2(x,t)$.

This is valid for materials which have their relaxation parameters defined within the range $0 < T_g < 1$. This may be particularly useful for nonviscous liquids where $T_1 \approx T_2 > 1.0$s. It may also be useful for the analysis of cerebrospinal fluid of $T_1 = 2.4$ s and $T_2 = 1.4$ s at 1.5 T.

In our current investigation, it is exciting and motivating to note that Eq. [5] is transformable to Bessel equation of order n and parameter α where

$$n = \frac{\tau}{T_1 T_2}$$

and $\alpha = \gamma G.\tau$

In this practical situation, $\Omega = \frac{1}{T_1 T_2} + \gamma^2 B_1^2(x)$, where $B_1(x)$ is independent of t, α is the NMR pulse,

G is a constant magnetic field gradient, τ is the interval between two pulses and γ is the gyromagnetic ratio. This assumption has allowed the use of the abundantly available and very rich Bessel functions and properties to address difficult NMR problems as will be presented in subsequent studies. Generally, the eight mathematical assumptions in this study are practical or experimental conditions that should be accurately defined for any NMR, MRI, or fMRI experiment.

CONCLUSION

Analytical models of the Bloch NMR flow equations ultimately cannot operate in a vacuum: the models must be used for solving problems, generally coming from outside of the restricted NMR/MRI experimental community. Often, however, the problems addressed end up requiring background in a number of fields rather than just in the sophisticated MRI workstations, and Laboratories mostly located in the advanced hospitals and oil companies. Fortunately, all or most of the problems solvable by NMR/MRI techniques are based on the fundamental Bloch NMR equations. These problems end up requiring additional knowledge of classical mechanics, graphics, mathematics, physics, chemistry, biology, and basic engineering knowledge. Typically, geometrical factors affect the radiographic image produced by MRI for medical and geophysical purposes, regardless of the format. Each clinical or geophysical study requires its own acquisition parameters to obtain the best quality suitable for the investigation. Despite over 50 years of the use of NMR/MRI for various investigations, the choice of technique parameters still relies to a great extent on experience. Scientific efforts to optimize the choice in terms of finding the parameter settings which yield sufficient image quality at the lowest possible cost are still rare. True optimization requires 1) estimation of the image quality needed to make a correct diagnosis and 2) methods to investigate all possible means of achieving this image quality in order to be able to decide which of them gives the lowest cost. These problems could be approached purely mathematically by solving the fundamental Bloch NMR flow equation analytically. As such it presents a significant challenge, either for a purely mathematical approach or as a minimization problem using successive approximations. In this presentation, calculations are based on the resolution of mathematical models in Cartesian, cylindrical, and spherical polar coordinates representing the geometrical questions of a typical magnetic resonance imaging method (*62–65*, *85–87*). Boundary conditions are involved in the resolution algorithm at different stages. Generally, we considered the transverse magnetization M_y, a function which describes the induced EMF in a typical NMR/MRI system. This function changes over time depending on the state of the sample based on the values of T_1 and T_2 relaxation parameters and the applied gradient. The boundary conditions are introduced based on Bessel functions in cylindrical coordinate and Bessel with Legendre polynomials in the case of spherical geometries (*69–74*, *79*). The advantage of these models is that, the Mathematician, Computer scientist, theoretical Physicist and the engineer can apply their tools and contribute to this fast developing and most exciting field of our time without acquiring the most sophisticated equipment.

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